AV-8870 Model Answer

B.Sc. (Hon's) (First Semester) Examination, 2015-16

Mathematics Paper-First (Calculus)

1. (i) $y = \frac{x}{(x-1)(x-2)}$ by partial fraction

$$y = \frac{-1}{x-1} + \frac{2}{x-2}$$

Using formula $D^n\{(ax+b)^{-1}\} = (-1)^n n!\, a^n\, (ax+b)^{-n-1}$

$$y_n = -(-1)^n n!\,(x-1)^{-n-1} + 2\,(-1)^n n!\,(x-2)^{-n-1}$$

$$= (-1)^n n! \left\{ \frac{2}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right\}$$ Ans.

(ii) Leibnitz's Theorem If u and v are two functions of x having differential coffiient of the nth order then nth differential coffiient of product of u and v ie uv is denoted by $D^n(uv)$ and given by

$$D^n(uv) = (D^n u).v + {}^nC_1 D^{n-1}u.Dv + {}^nC_2 D^{n-2}u.D^2v + \cdots$$
$$+ \cdots {}^nC_r D^{n-r}u.D^r v + \cdots + u.D^n v$$

(iii) $f(x)$ is said to be differentiable at $x=c$ if

$$Lf'(c) = Rf'(c) = f'(c)$$

ie $$\lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f'(c)$$

(iv) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Diff. w.r.t x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}\frac{b^2}{a^2}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{x_1}{y_1}\frac{b^2}{a^2}$$

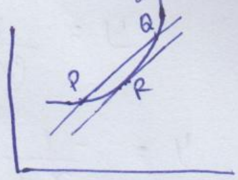
Equation of Normal is $(y-y_1)\left(\frac{dy}{dx}\right)_{(x_1,y_1)} + (x-x_1) = 0$

$$(y-y_1)\left(-\frac{b^2x_1}{a^2y_1}\right) + (x-x_1) = 0$$

$$a^2y_1(x-x_1) - b^2x_1(y-y_1) = 0$$ Ans.

(vi) <u>Geometrical interpretation of Lagrange's Mean Value Theorem</u>

Lagrange's mean value theorem geometrically asserts that there is some point between P and Q where the tangent is parallel to PQ.



(vi) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ (To show)

Using ϵ-δ definition if $\lim_{x \to 0} \left(x \sin\frac{1}{x} = 0\right)$ then

$\forall \epsilon > 0, \ \exists \delta > 0 \quad |x \sin \frac{1}{x} - 0| < \epsilon$

ie $|x \sin \frac{1}{x}| < \epsilon$

Consider $|x \sin\frac{1}{x}| = |x| \cdot |\sin \frac{1}{x}|$

$\because \sin x$ is bounded and $|\sin x| \leq 1$

$\Rightarrow \quad |x \sin \frac{1}{x}| \leq |x| \cdot 1$ —— ①

Take $\delta = \epsilon$ If $|x - 0| < \delta$ ie $|x| < \delta$ and $\delta = \epsilon$

$\Rightarrow |x| < \epsilon$ using ①

$|x \sin\frac{1}{x}| \leq |x| < \epsilon \Rightarrow |x \sin\frac{1}{x}| < \epsilon \quad \forall \ |x| < \delta$

hence we shown $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

(vii) $f(x) = 2x^3 + 7x^2 + x - 1$

Taylor's Theorem $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) \ldots$

Here $f(x) = 2x^3 + 7x^2 + x - 1$ $\qquad f(2) = 2 \cdot 2^3 + 7 \cdot 2^2 + 2 - 1 = 45$

$f'(x) = 6x^2 + 14x + 1$ $\qquad f'(2) = 6 \cdot 2^2 + 14 \cdot 2 + 1 = 53$

$f''(x) = 12x + 14$ $\qquad f''(2) = 12 \cdot 2 + 14 = 38$

$f'''(x) = 12$ $\qquad f'''(2) = 12$

Hence $2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + \frac{(x-2)^2}{2!} 38 + \frac{(x-2)^3}{3!} \cdot 12$

$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

(viii) $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2} \quad \left(\frac{0}{0}\right)$

Using L's Hospital Rule

$= \lim_{x \to 0} \frac{e^x - e^{-x}}{2x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{e^x + e^{-x}}{2}$

$= 1$ <u>Ans.</u>

2. $y = \sin(m\sin^{-1}x)$

Diff. w.r.t. x

$y_1 = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$

$(1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x) = m^2(1-y^2)$

Diff. w.r.t. x

$(1-x^2)\,2y_1 \cdot y_2 + y_1^2(-2x) = m^2(-2y\,y_1)$

$(1-x^2)y_2 - xy_1 = -m^2 y$

$(1-x^2)y_2 - xy_1 + m^2 y = 0$

Using Leibnitz's Theorem

$D^n\{(1-x^2)y_2\} = y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)$

$D^n\{xy_1\} = y_{n+1}\cdot x + n\,y_n \cdot 1$

$D^n\{m^2 y\} = m^2 y_n$

$(1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - x\,y_{n+1} - n\,y_n + m^2 y_n = 0$

$(1-x^2)y_{n+2} = (2n+1)x\,y_{n+1} + (n^2-m^2)y_n$ #

Put $x=0$ $\quad y_0 = 0$

$(y_1)_0 = m$

$(y_2)_0 = 0$

$(y_{n+2})_0 = (n^2-m^2)(y_n)_0$

Put $n=1 \quad (y_3)_0 = (1^2-m^2)\cdot(y_1)_0 = (1^2-m^2)\cdot m$

$n=2 \quad (y_4)_0 = (2^2-m^2)\cdot(y_2)_0 = 0$

$n=3 \quad (y_5)_0 = (3^2-m^2)\cdot(y_3)_0 = (3^2-m^2)\cdot(1^2-m^2)\cdot m$

$n=4 \quad (y_6)_0 = (4^2-m^2)\cdot(y_4)_0 = 0$

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If $n=$ even $\quad (y_n)_0 = 0$

$n=$ odd $\quad (y_n)_0 = \{(n-2)^2-m^2\}\{(n-4)^2-m^2\} - - - (3^2-m^2)(1^2-m^2)\cdot m$ } Ans.

3. $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x & 0 \le x \le 1/2 \\ 1-x & 1/2 \le x \le 1 \end{cases}$$

Check the continuity at $x = 1/2$

L.H.L. at $x=1/2$ $\quad \lim_{x \to 1/2^-} f(x) = \lim_{h \to 0} f(1/2 - h) = \lim_{h\to 0} 1/2 - h = 1/2$

R.H.L. at $x = 1/2$ $\quad \lim_{x \to 1/2^+} f(x) = \lim_{h \to 0} f(1/2 + h) = \lim_{h \to 0} 1 - 1/2 - h = 1/2$

and $f(1/2) = 1/2$

$\because$ L.H.L. = R.H.L. = $f(1/2)$ $\Rightarrow$ $f(x)$ is continuous at $x = 1/2$.

Check the differentiability at $x = 1/2$.

$$Lf'(1/2) = \lim_{h \to 0} \frac{f(1/2 - h) - f(1/2)}{-h} = \lim_{h \to 0} \frac{\cancel{1/2} - h - \cancel{1/2}}{-h} = 1$$

$$Rf'(\tfrac{1}{2}) = \lim_{h \to 0} \frac{f(1/2 + h) - f(1/2)}{h} = \lim_{h \to 0} \frac{1 - 1/2 - h - 1/2}{h} = -1$$

$\because$ $Lf'(1/2) \ne Rf'(1/2)$ hence $f(x)$ is not differentiable at $x = 1/2$.

4. Tracing of $y^2(a+x) = x^2(3a - x)$

(I) Symmetry. Powers of y are even $\Rightarrow$ Curve is symmetric along X Axis

(II) Passing through origin There is no constant term so curve will pass through origin.

Nature at origin Lowest degree term = 0

$$ay^2 - 3ax^2 = 0$$

$$y = \pm \sqrt{3}\, x \quad \text{two real tangents}$$

$\Rightarrow$ Origin is node for given Curve

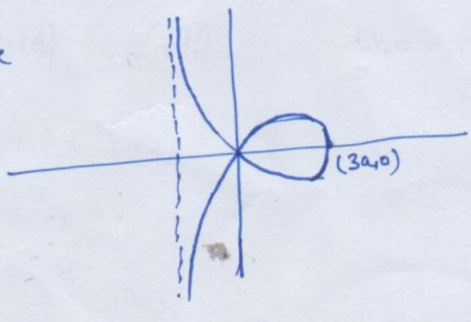
(III) Intersection with X Axis : $y = 0$ $\Rightarrow$ $x = 0, 3a$

Pt. $(0,0)$ $(3a, 0)$

Y Axis : $x = 0$ $\Rightarrow$ $y = 0$ $\Rightarrow$ $(0,0)$

(IV) Asymptote $\parallel$ to $\cancel{X}$ Axis : $a + x = 0$ $\Rightarrow$ $x = -a$

$\parallel$ to X Axis : $1 \ne 0$ No Asymptote



(V) Range of curve $\quad y = \dfrac{x\sqrt{3a - x}}{\sqrt{a + x}}$

$x > 3a$ $\quad y$ = imaginary

$x < -a$ $\quad y$ = imaginary

5. $r^n = a^n \cos n\theta$

Diff. w.r.t. θ

$\cancel{n} r^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta \cdot \cancel{n} \Rightarrow \frac{dr}{d\theta} = \frac{-a^n \sin n\theta \cdot r}{r^n} = -\frac{\cancel{a^n} \cdot r \sin n\theta}{\cancel{a^n} \cos n\theta} = -r \tan n\theta$

$\frac{d^2r}{d\theta^2} = -r \cdot \sec^2 n\theta \cdot n - \tan n\theta \cdot \frac{dr}{d\theta} = -nr\sec^2 n\theta + r\tan^2 n\theta$

Radius of curvature $\rho = \dfrac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$

$= \dfrac{\{r^2 + r^2\tan^2\theta\}^{3/2}}{r^2 + 2r^2\tan^2\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$

$= \dfrac{r^3 \sec^3 n\theta}{r^2 \cdot \sec^2 n\theta + nr^2\sec^2 n\theta} = \dfrac{r \sec n\theta}{(n+1)} = \dfrac{r}{(n+1)\cos n\theta} = \dfrac{r}{(n+1) \cdot r^n a^{-n}} = \dfrac{a^n}{(n+1) r^{n-1}}$

6. $\lim_{x \to 0} \dfrac{\sin 2x - a\sin x}{x^3} \quad \left(\frac{0}{0}\right)$

Using L' Hospital's Rule

$= \lim_{x \to 0} \dfrac{2\cos 2x - a\cos x}{3x^2}$

Above limit tends to a finite limit only if $\lim_{x \to 0} 2\cos 2x - a\cos x = 0$

$\Rightarrow 2 - a = 0 \Rightarrow \boxed{a = 2}$

then above limit will be

$= \lim_{x \to 0} \dfrac{2\cos 2x - 2\cos x}{3x^2} \quad \left(\frac{0}{0}\right)$

Again Applying L' Hospital's Rule

$= \lim_{x \to 0} \dfrac{-4\sin 2x + 2\sin x}{6x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \dfrac{-8\cos 2x + 2\cos x}{6}$

$= \dfrac{-8+2}{6} = -1$

Value of limit $= -1$ Ans

7. $y = f(x) = x^5 - 5x^4 + 5x^3 - 1$

Necessary Condition for maxima/minima of $f(x)$

is $\frac{dy}{dx} = 0 \Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$

$5x^2(x^2 - 4x + 3) = 0$

$5x^2(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3.$

$\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

At $x = 1$ $\frac{d^2y}{dx^2} = 20 - 60 + 30 = -10 < 0$ $\Rightarrow x = 1$ is Point of Maxima

$x = 3$ $\frac{d^2y}{dx^2} = 20(3)^3 - 60(3)^2 + 30(3) = 90 > 0$ $\Rightarrow x = 3$ is Point of Minima

$x = 0$ $\frac{d^2y}{dx^2} = 0$ then $\frac{d^3y}{dx^3} = 60x^2 - 120x + 30$ at $x = 0$ $\frac{d^3y}{dx^3} = 30 \neq 0$ $\Rightarrow x = 0$ is Point of neither Maxima nor Minima

[8] $y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x-1=0$

Put $y=mx+c$

$$(mx+c)^3-5x(mx+c)^2+8x^2(mx+c)-4x^3-3(mx+c)^2+9x(mx+c)$$
$$-6x^2+2(mx+c)-2x-1=0$$

$$(m^3-5m^2+8m-4)x^3+(3m^2c-10mc+8c-3m^2+9m-6)x^2+(3mc^2-5c^2-6mc+9c+2m-2)x+(--)=0$$

For Asymptote equating coff. of $x^3=0$

$\Rightarrow m^3-5m^2+8m-4=0$

$\Rightarrow m^2(m-1)-4m(m-1)+4(m-1)=0$

$\Rightarrow (m-1)(m-2)^2=0 \quad \Rightarrow \quad m=1,2,2$

Coff. of $x^2=0 \quad \Rightarrow (3m^2-10m+8)c-3m^2+9m-6=0$

$\Rightarrow c=\dfrac{3m^2-9m+6}{3m^2-10m+8}$

For $m=1$ $\quad c=\dfrac{3-9+6}{3-10+8}=0$

$m=2$ $\quad c=\dfrac{12-18+6}{12-20+8}$ Not determinable

hence put coff of $x=0 \quad \Rightarrow \quad 3mc^2-5c^2-6mc+9c+2m-2=0$

For $m=2$ $\quad 6c^2-5c^2-12c+9c+4-2=0$

$c^2-3c+2=0$

$c=\dfrac{3\pm\sqrt{9-8}}{2}=\dfrac{3\pm1}{2}=2,1$

$m=1, \quad c=0$ Corresponding Asymptote $y=x$

$m=2, \quad c=2$ " " $y=2x+2$

$m=2, \quad c=1$ " " $y=2x+1$

$\left.\begin{array}{l} y=x \\ y=2x+2 \\ y=2x+1 \end{array}\right\}$ 3 Asymptotes Ans.